

Nonplanar Periodic Solutions for Spatial Restricted N+1-Body Problems *

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Abstract: We use variational minimizing methods to study spatial restricted N+1-body problems with a zero mass moving on the vertical axis of the moving plane for N equal masses. We prove that the minimizer of the Lagrangian action on the anti-T/2 or odd symmetric loop space must be a non-planar periodic solution for any $N \geq 2$.

Keywords: Restricted N+1-body problems; nonplanar periodic solutions; variational minimizers; Jacobi's necessary conditions.

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1 Introduction and Main Result

Spatial restricted 3-body model was studied by Sitnikov [5]. Mathlouthis [3] etc. studied the periodic solutions for the spatial circular restricted 3-body problems by minimax variational methods.

In this paper, we study spatial circular restricted N+1-body problems with a zero mass moving on the vertical axis of the moving plane for N equal masses. Suppose point masses $m_1 = \dots = m_N = 1$ move centered at the center of masses on a circular orbit. The motion for the zero mass is governed by the gravitational forces of m_1, \dots, m_N . Let $\rho_j = e^{\sqrt{-1}\frac{2\pi j}{N}}$ and

$$q_1(t) = re^{\sqrt{-1}2\pi t}\rho_1, \dots, q_j(t) = \rho_j q_1(t), \dots, q_N(t) = re^{\sqrt{-1}2\pi t} \quad (1.1)$$

satisfy the Newtonian equations:

$$m_i \ddot{q}_i = \frac{\partial U}{\partial q_i}, \quad i = 1, \dots, N, \quad (1.2)$$

where

$$U = \sum_{1 \leq i < j \leq N} \frac{m_i m_j}{|q_i - q_j|}. \quad (1.3)$$

The orbit $q(t) = (0, 0, z(t)) \in R^3$ for zero mass satisfies the following equation

$$\ddot{q} = \sum_{i=1}^N \frac{m_i(q_i - q)}{|q_i - q|^3}. \quad (1.4)$$

Define

$$f(q) = \int_0^1 \left[\frac{1}{2} |\dot{q}|^2 + \sum_{i=1}^N \frac{1}{|q - q_i|} \right] dt, \quad q \in \Lambda_i, \quad (1.5)$$

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then

$$f(q) = \int_0^1 \left[\frac{1}{2} |z'|^2 + \frac{N}{\sqrt{r^2 + z^2}} \right] dt \triangleq f(z), \quad q \in \Lambda_i, \quad (1.6)$$

where

$$\begin{aligned} \Lambda_1 &= \left\{ \begin{array}{l} q(t) = (0, 0, z(t)) | z(t) \in W^{1,2}(R/Z, R) \\ z(t + \frac{1}{2}) = -z(t), \quad q(t) \neq q_i(t), \quad \forall t \in R, i = 1, 2, \dots, N \end{array} \right\}, \\ \Lambda_2 &= \left\{ \begin{array}{l} q(t) = (0, 0, z(t)) | z(t) \in W^{1,2}(R/Z, R) \\ q(-t) = -q(t) \end{array} \right\}, \\ W^{1,2}(R/Z, R) &= \left\{ x(t) \left| \begin{array}{l} x(t), \dot{x}(t) \in L^2(R, R) \\ x(t+1) = x(t) \end{array} \right. \right\}. \end{aligned}$$

Notice that the symmetry in Λ_1 is related with Italian symmetry [1].

In this paper, our main result is the following:

Theorem 1.1 The minimizer of $f(q)$ on the closure $\bar{\Lambda}_i$ of Λ_i ($i=1,2$) is a nonplanar and noncollision periodic solution.

2 Proof of Theorem 1.1

We define the inner product and equivalent norm of $W^{1,2}(R/Z, R)$:

$$\langle u, v \rangle = \int_0^1 (uv + u' \cdot v') dt, \quad (2.1)$$

$$\begin{aligned} \|u\| &= \left[\int_0^1 |u|^2 dt \right]^{\frac{1}{2}} + \left[\int_0^1 |u'|^2 dt \right]^{\frac{1}{2}} \\ &\cong \left[\int_0^1 |u'|^2 dt \right]^{\frac{1}{2}} + |u(0)|. \end{aligned} \quad (2.2)$$

Lemma 2.1 (Palais's Symmetry Principle ([4])) Let σ be an orthogonal representation of a finite or compact group G in the real Hilbert space H such that for $\forall \sigma \in G, f(\sigma \cdot x) = f(x)$, where $f : H \rightarrow R$.

Let $S = \{x \in H | \sigma \cdot x = x, \forall \sigma \in G\}$. Then the critical point of f in S is also a critical point of f in H .

By Palais's Symmetry Principle, we know that the critical point of $f(q)$ in $\bar{\Lambda}_i$ is a noncollision periodic solution of Newtonian equation (1.4).

In order to prove Theorem 1.1, we need

Lemma 2.2 ([6]) Let X be a reflexive Banach space, S be a weakly closed subset of X , $f : S \rightarrow R \cup +\infty$, $f \not\equiv +\infty$ is weakly lower semi-continuous and coercive ($f(x) \rightarrow +\infty$ as $\|x\| \rightarrow +\infty$), then f attains its infimum on S .

Lemma 2.3 (Poincare-Wirtinger Inequality) Let $q \in W^{1,2}(R/Z, R^N)$ and $\int_0^T q(t) dt = 0$, then

$$\int_0^T |\dot{q}(t)|^2 dt \geq \left(\frac{2\pi}{T} \right)^2 \int_0^T |q(t)|^2 dt. \quad (2.3)$$

Lemma 2.4 $f(q)$ in (1.6) attains its infimum on $\bar{\Lambda}_1 = \Lambda_1$ or $\bar{\Lambda}_2 = \Lambda_2$.

Proof. By Lemma 2.2 and Lemma 2.3, it is easy to prove Lemma 2.4.

Lemma 2.5(Jacobi's Necessary Condition[2]) If the critical point $u = \tilde{u}(t)$ corresponds to a minimum of the functional $\int_a^b F(t, u(t), u'(t))dt$ and if $F_{u'u'} > 0$ along this critical point, then the open interval (a, b) contains no points conjugate to a , that is, for $\forall c \in (a, b)$, the following boundary value problem:

$$\begin{cases} -\frac{d}{dt}(Ph') + Qh = 0, \\ h(a) = 0, \quad h(c) = 0, \end{cases} \quad (2.4)$$

has only the trivial solution $h(t) \equiv 0$, $\forall t \in (a, c)$, where

$$P = \frac{1}{2}F_{u'u'}|_{u=\tilde{u}}, \quad (2.5)$$

$$Q = \frac{1}{2}(F_{uu} - \frac{d}{dt}F_{uu'})|_{u=\tilde{u}}. \quad (2.6)$$

Lemma 2.6 The radius r for the moving orbit of N equal masses is

$$r = \left(\frac{1}{4\pi}\right)^{\frac{2}{3}} \left[\sum_{1 \leq j \leq N-1} \csc\left(\frac{\pi}{N}j\right) \right]^{\frac{1}{3}}.$$

Proof. By (1.1)-(1.3), we have

$$\ddot{q}_N = \sum_{j \neq N} \frac{q_j - q_N}{|q_j - q_N|^3}, \quad (2.7)$$

Substituting (1.1) into (2.7), we have

$$-4\pi^2 = \sum_{j \neq N} \frac{\rho_j - \rho_N}{r^3 |\rho_j - \rho_N|^3} \quad (2.8)$$

$$\begin{aligned} 4\pi^2 r^3 &= \sum_{j \neq N} \frac{1 - \rho_j}{|1 - \rho_j|^3} \\ &= \frac{1}{4} \sum_{1 \leq j \leq N-1} \csc\left(\frac{\pi}{N}j\right) \end{aligned} \quad (2.9)$$

Then

$$r^3 = \frac{1}{16\pi^2} \sum_{1 \leq j \leq N-1} \csc\left(\frac{\pi}{N}j\right). \quad (2.10)$$

Therefore

$$r = \left(\frac{1}{4\pi}\right)^{\frac{2}{3}} \left[\sum_{1 \leq j \leq N-1} \csc\left(\frac{\pi}{N}j\right) \right]^{\frac{1}{3}}. \quad (2.11)$$

Lemma 2.7([8]) $\sum_{j=1}^{N-1} \csc\left(\frac{\pi}{N}j\right) = \frac{4}{N}$.

For the functional (1.6), let

$$F(z, z') = \frac{1}{2}|z'|^2 + \frac{N}{\sqrt{r^2 + z^2}}.$$

Then the second variation of (1.6) in the neighborhood of $z = 0$ is given by

$$\int_0^1 (Ph'^2 + Qh^2)dt, \quad (2.12)$$

where

$$P = \frac{1}{2}F_{z'z'}|_{z=0} = \frac{1}{2}, \quad (2.13)$$

$$Q = \frac{1}{2}(F_{zz} - \frac{d}{dt}F_{zz'})|_{z=0} = -\frac{N}{2r^3}. \quad (2.14)$$

The Euler equation of (2.12) is called the Jacobi equation of the original functional (1.6), which is

$$-\frac{d}{dt}(Ph'^2) + Qh = 0, \quad (2.15)$$

That is,

$$h'' + \frac{N}{r^3}h = 0. \quad (2.16)$$

Next, we study the solution of (2.16) with initial values $h(0) = 0$, $h'(0) = 1$. It is easy to get

$$h(t) = \sqrt{\frac{r^3}{N}} \cdot \sin \sqrt{\frac{N}{r^3}}t, \quad (2.17)$$

which is not identically zero on $[0, \frac{1}{2}]$, but we will prove $h(\frac{1}{2}) = 0$, and $h(c) = 0$ for some $c \in (0, \frac{1}{2})$. Notice that

$$\sqrt{\frac{N}{r^3}} = \sqrt{N}4\pi \left(\sum_{j \neq N} \csc \frac{\pi}{N}j \right)^{-\frac{1}{2}} \quad (2.18)$$

Hence

$$\begin{aligned} \frac{1}{2}\sqrt{\frac{N}{r^3}} &= \sqrt{N} \left(\sum_{j \neq N} \csc \frac{\pi}{N}j \right)^{-\frac{1}{2}} \cdot 2\pi \\ &= \sqrt{N} \left(\frac{4}{N} \right)^{-\frac{1}{2}} \cdot 2\pi \\ &= N\pi. \end{aligned} \quad (2.19)$$

So

$$h(\frac{1}{2}) = 0. \quad (2.20)$$

Given $N \geq 2$, choose $0 < c = \frac{1}{2N} < \frac{1}{2}$ such that $2Nc = 1$, then

$$\sqrt{\frac{N}{r^3}}c = 2N\pi c = \pi \quad (2.21)$$

Therefore

$$\sin \sqrt{\frac{N}{r^3}}c = \sin \pi = 0. \quad (2.22)$$

Hence $q(t) = (0, 0, 0)$ is not a local minimum for $f(q)$ on $\bar{\Lambda}_i = \Lambda_i (i = 1, 2)$. So the minimizers of $f(q)$ on Λ_i are not always at the center of masses, they must oscillate periodically on the vertical axis, that is, the minimizers are not always co-planar, hence we get the non-planar periodic solutions.

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